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Method of Dimensionality Reduction in Contact Mechanics: A Simple Engineering Tool for Applications to Complex Geometries and Gradient Materials

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1. Introduction

From the perspective of development of effective modeling techniques, the history of science is the story of "reduction" in the broadest sense, finally leading to the reduction of the complexity of the physical models: reduction of degrees of freedom, reduction of dimensionality, data compression during its generation, storage and processing. The reduction of complexity is often achieved by a change in "looking at things". From the mathematical point of view this often means just a change in the "parameterization" of the system. The effectiveness of reducing the complexity of the system is determined by the quality of the "test functions" used to represent solutions. For example, Fourier analysis, which is known as one of the most powerful methods for both analytical and numerical studies of systems described by partial differential equations, is based on the change of parameterization by coordinates through the parameterization by the wave vectors. Similar ideas underlie the "modal" reduction in the structural dynamics. Of course, an effective parameterization depends on the problem to be solved. Therefore, no "universal" reduction method does exist that can be used in any situation.

A look at the history of science shows that in many cases a simple change of the parameterization lead to a breakthrough. Thus, the finite element method has won its position of the most important method in simulation technology due to the parameterization directly through the nodal variables. Another very popular and powerful idea used both in computer technology and biological systems (brain) is to use a hierarchical reduction. To a certain extent, this is a generalization of the Fourier analysis. Hierarchical reduction is known to anyone using Google maps. It allows to "zoom in and out" the images and to consider them with various magnification. In the contact mechanics there is a series of simulation methods based on hierarchical reduction as e.g. the multilevel Lubrecht concept [1] [2], [3] as well as the analytical contact mechanics of randomly rough surfaces, which have been developed by Persson [4].

An important class of contact problems is the "one-contact" or "single asperity" problem. It appears in the measurement of indentation hardness, and is the basis for further generalizations to multi-contact systems. For the single asperity problem, there exists an extremely simple and effective solution which is based on the ideas of Jäger [5]. He suggested using as "basic functions" the solutions of the indentation problem of a rigid cylinder with variable radius. In connection with the superposition principle of Cattaneo [6] and Mindlin [7] which allows reduction of tangential contact problem to a normal one and the principle of functional equations of Lee [8] and Radok [9] allowing reduction of a viscoelastic contact to the elastic one, as well as reduction the problem of adhesive contact to the superposition of two solutions for non-adhesive contacts (JKR theory [10]), the Jäger superposition principle allows formulating an extremely powerful concept for solving the one asperity problems - the so-called Method of Dimensionality Reduction (MDR) [11], [12].

In this paper we provide a brief overview of the main procedures and applications fields of this method.

2. Jäger superposition principle

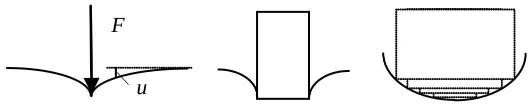
Most of the methods for solving contact problems are based on the so-called "fundamental solution" of elasticity theory, which determines the deformation of the contact under the action of a force. For the isotropic elastic half-space, this solution has the form [13]

$$u_z = \frac{1}{\pi E^*}\frac{1}{r}F \ , \qquad (1)$$

where u_z is the normal displacement of a surface point, F is the normal force, r the polar radius in the contact surface, and E^* the effective elastic modulus. An arbitrary stress distribution $p(x',y')$ then leads to the surface deformation

$$u_z = \frac{1}{\pi E^*}\iint p(x',y')\frac{dx'dy'}{r} \ , \quad r = \sqrt{(x-x')^2+(y-y')^2} \ . \qquad (2)$$

This equation is the basis both for almost all analytical solutions of contact problems and for the numerical solution via Boundary Element Method (BEM) [14]. This way of formulating and solution of contact problems is possible for any media and geometrical configurations for which the fundamental solution is known. Another way is to use a superposition of indentations by cylindrical punches (Fig. 1b,c).



a b c

Fig. 1 (a) Fundamental solution is a basis of all "standard" formulations of analytical and numerical simulation methods in contact mechanics. (b) Indentation of a flat-ended punch. (c) Jäger representation of arbitrary axis-symmetric body as superposition of flat-ended cylindrical indentations.

If the penetration depth d is known as a function of the radius a of the contact:

$$d = g(a) \ , \qquad (3)$$

then, the normal force F_N can be represented as a function of penetration depth by the trivial equation

$$F_N = \int_0^{F_N} \mathrm{d}\tilde{F}_N = \int_0^a \frac{\mathrm{d}\tilde{F}_N}{\mathrm{d}\tilde{d}}\frac{\mathrm{d}\tilde{d}}{\mathrm{d}\tilde{a}}\mathrm{d}\tilde{a} = \int_0^a k(\tilde{a})\frac{\mathrm{d}g(\tilde{a})}{\mathrm{d}\tilde{a}}\mathrm{d}\tilde{a}$$
$$= \int_0^a \frac{\mathrm{d}k(\tilde{a})}{\mathrm{d}\tilde{a}}\left(d - g(\tilde{a})\right)\mathrm{d}\tilde{a} \qquad (4)$$

where $k(\tilde{a})$ is stiffness of a cylindrical punch with radius $\tilde{a}$.

Remarkable feature of the equations (3) and (4) it is that they can be interpreted as the result of the indentation of the modified profile $g(\tilde{a})$ in the elastic foundation with independent springs with spacing $\mathrm{d}\tilde{a}$ and stiffness $\frac{1}{2}\frac{\mathrm{d}k(\tilde{a})}{\mathrm{d}\tilde{a}}\mathrm{d}\tilde{a}$. In accordance with the equation (4), the use of MDR is possible under two conditions: (1) the contact stiffness $k(\tilde{a})$ of a cylindrical die with a radius $\tilde{a}$ must be known and (2) the rule of determining the modified profile $g(\tilde{a})$ is known. It does not matter how these two steps are performed: analytically, numerically or experimentally.

For homogeneous media, these rules are known explicitly. If the initial three-dimensional profile is $f(r)$ then the MDR-transformed profile is determined by the Eq.

$$g(x) = |x| \int_0^{|x|} \frac{f'(r)}{\sqrt{x^2 - r^2}} \mathrm{d}r \ . \tag{5}$$

The stiffness of a contact with a cylindrical punch is $k(\tilde{a}) = 2E^* a$, thus providing the rule for the stiffness of springs of equivalent elastic foundation

$$\mathrm{d}k = E^* \mathrm{d}x \ . \tag{6}$$

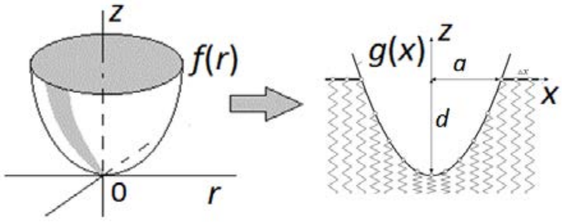


Fig. 2 MDR-transformation: The initial profile is replaced by a transformed one and at the same time the elastic half-space by the equivalent Winkler foundation.

After the solution of the equivalent one-dimensional problem is obtained three-dimensional exact solution can be restored with the help of simple rules [11]. The solution of the MDR formally reproduces the solution first obtained by Galin [15] and later used by Sneddon in his very much cited paper [16].

3. MDR applications: adhesive, tangential, viscoelastic contacts and contacts with gradient materials

The main advantage of the MDR is that it allows solving of many related tasks within a single formalism. Using the same transformation (5), we can consider the following tasks: tangential contact in the approximation of Cattaneo-Mindlin [17], adhesive contact [17], and contacts with the viscoelastic bodies [18]. In the case of contacts of functionally graded materials it is necessary to use another transformation [19], [20].

In the following, we give a list of problems which can be handled using the MDR:
- Normal contact of arbitrary shapes
- Tangential contact of arbitrary shapes
- Rolling contact [21], [22]
- Adhesive contacts of rotationally symmetric shapes
- Heat transfer and generation in frictional contacts
- Electrical conduction (arbitrary shapes)
- Contacts with viscoelastic media (arbitrary shapes)
- Contacts with gradient media
- Frictional damping (arbitrary shapes and loading histories)
- Multiscale roughness
- Wear calculation for rotationally symmetric shapes
- Torsional contact [23]

4. Conclusion: Why the Winkler foundation is so reliable?

We now understand why the Winkler foundation, which has been used over decades in numerous applications for *qualitative* analysis of contact problems, does work so amazingly well. The MDR shows that in many cases it provides even *exact* solutions: one just has to follow the rules of the Method of Dimensionality Reduction.

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